

# THE CUBICAL HOMOLOGY OF TRACE MONOIDS

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## Abstract

This article contains an overview of the results of the author in a field of algebraic topology used in computer science. The relationship between the cubical homology groups of generalized tori and homology groups of partial trace monoid actions is described. Algorithms for computing the homology groups of asynchronous systems, Petri nets, and Mazurkiewicz trace languages are shown.

Keywords: semicubical set, homology of small categories, free partially commutative monoid, trace monoid, asynchronous transition system, Petri nets, trace languages.

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## Introduction

Trace monoids have found many applications in computer science [17], [3]. M. Bednarczyk [2] studied and applied the category of asynchronous systems. The author has proved that any asynchronous system can be regarded as a partial trace monoid with action on a set. It is allowed to build homology theory for the category of asynchronous systems and Petri nets [9]. In this paper we also introduce a homology for Mazurkiewicz trace languages. It should be noted that the homology theory was introduced and studied for higher dimensional automata in the works [7] and [6]. E. Haucourt [8] applied the Baues-Wirsching homology.

We study of a relationship between the cubical homology of generalized tori and homology of a trace monoid action on a set. We build the algorithms for the computing the homology groups of asynchronous systems, elementary Petri nets, and Mazurkiewicz trace languages. It allows us to solve the problem posed in [9, Open problem 1] of the constructing an algorithm for computing homology groups of the elementary Petri nets.

# 1 Trace monoids and their partial actions

This section is devoted to the basic definitions and the problems that have appeared.

## 1.1 Notations

We first describe our notations. Let  $\mathbf{Set}$  be a category of sets and maps and let  $\mathbf{Ab}$  be a category of Abelian groups and homomorphisms. Denote by  $\mathbb{Z}$  the set of the additive group of integers. Let  $\mathbb{N}$  be the set of nonnegative integers or the free monoid  $\{1, a, a^2, \dots\}$  generated by one element. Given a category  $\mathcal{A}$  denote by  $\mathcal{A}^{op}$  the opposite category. Denote by  $\mathbf{Ob} \mathcal{A}$  the class of all objects and  $\mathbf{Mor} \mathcal{A}$  the class of all morphisms in the category  $\mathcal{A}$ . Given objects  $a, b \in \mathbf{Ob} \mathcal{A}$  denote by  $\mathcal{A}(a, b)$  the set of all morphisms  $a \rightarrow b$ . For any small category  $\mathcal{C}$ , functors  $F : \mathcal{C} \rightarrow \mathcal{A}$  will be called *diagrams of objects in  $\mathcal{A}$  on  $\mathcal{C}$* . In this case, along with the notation  $F : \mathcal{C} \rightarrow \mathcal{A}$  we use the notation  $\{F(c)\}_{c \in \mathcal{C}}$ . The category  $\mathcal{A}^{\mathcal{C}}$  of functors  $\mathcal{C} \rightarrow \mathcal{A}$  is called to be a *diagram category*.

Let  $\Delta \mathbb{Z} : \mathcal{C} \rightarrow \mathbf{Ab}$  be the diagram having the value  $\mathbb{Z}$  at each  $c \in \mathbf{Ob} \mathcal{C}$  and the value  $1_{\mathbb{Z}}$  at each  $\alpha \in \mathbf{Mor} \mathcal{C}$ .

Given a family of Abelian groups  $\{A_j\}_{j \in J}$  the direct sum is denoted by  $\bigoplus_{j \in J} A_j$ . Elements of summands are denoted as pairs  $(j, g)$  with  $j \in J$  and  $g \in A_j$ . If  $A_j = A$  for all  $j \in J$ , then this direct is denoted  $\bigoplus_{j \in J} A$ . If instead of a set  $J$  indicated a cardinal number  $p = |J|$ , then the direct coproduct is denoted by  $A^{(p)}$ .

## 1.2 Trace monoids

Let  $E$  be a set and let  $I \subseteq E \times E$  be an arbitrary subset. The set  $I \subseteq E \times E$  is an *independence relation* on  $E$  if the following conditions are satisfied:

- $(\forall a \in E)(a, a) \notin I$ ,
- $(\forall a \in E)(\forall b \in E) (a, b) \in I \Rightarrow (b, a) \in I$ .

Let  $E^*$  be the free monoid generated by a set  $E$ . It consists of the words in alphabet  $E$ . The binary operation is defined as the concatenation of words  $(a_1 \cdots a_m, b_1 \cdots b_n) \mapsto a_1 \cdots a_m b_1 \cdots b_n$ . The empty word is denoted by 1.

**DEFINITION 1.1** *Let  $I$  be an independence relation on a set  $E$ . A trace monoid (or free partially commutative monoid)  $M(E, I)$  is the factor monoid  $E^*/(\equiv)$  by a least equivalence relation for which  $uabv \equiv ubav$ , for all  $(a, b) \in I$ ,  $u \in E^*$ ,  $v \in E^*$ . Elements  $a, b \in E$  for which  $(a, b) \in I$  are called commuting generators.*

This definition is more general than given in [3] since we do not demand that  $E$  is finite.

For example, if  $E = \{a, b\}$ ,  $I = \{(a, b), (b, a)\}$ , then  $M(E, I) \cong \mathbb{N}^2$  is the free commutative monoid generated by two elements.

If  $I = \emptyset$ , then  $M(E, I) = E^*$ .

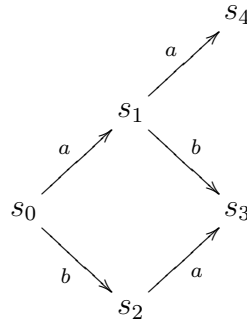
Any element  $w = a_1 \cdots a_n \in M(E, I)$  of a trace monoid can be interpreted as finite sequence of instructions  $a_1, a_2, \dots, a_n$  in a program. The relation  $I$  consists of pairs  $(a, b)$  instructions which can be executed concurrently.

### 1.3 State space

A *partial map*  $f : E \rightarrow E'$  between sets  $E$  and  $E'$  is a relation  $f \subseteq E \times E'$  for which  $(e, e'_1) \in f$  &  $(e, e'_2) \in f$  implies  $e'_1 = e'_2$ . Let  $PSet$  be the category of sets and partial maps between them. Any trace monoid  $M(E, I)$  can be considered as a category with the unique object denoted by  $o(M(E, I))$ .

A *partial trace monoid action* of  $M(E, I)$  on a set  $S$  is a functor  $\mathbf{S} : M(E, I)^{op} \rightarrow PSet$  such that its value at  $o(M(E, I))$  equals  $S$ . We denote  $\mathbf{S}(w)(s)$  by  $s \cdot w$ . A *state space*  $(M(E, I), S)$  consists of a trace monoid  $M(E, I)$  with a partial action on a set  $S$ . A state space  $(M(E, I), S)$  is determined by partial maps  $(-) \cdot a : S \rightarrow S$  corresponding to  $a \in E$ . Hence, it can be given by a directed graph with vertexes  $s \in S$  and labeled edges  $s \xrightarrow{a} s \cdot e$ .

For example, if  $E = \{a, b\}$  and  $I = \{(a, b), (b, a)\}$ , then the directed graph with labeled edges

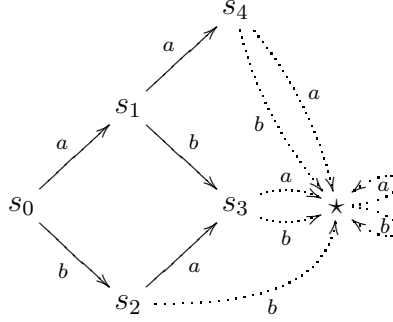


determines the action for which  $s_0 \cdot a = s_1$ ,  $s_0 \cdot b = s_2$ ,  $s_1 \cdot a = s_4$ ,  $s_1 \cdot b = s_3$ ,  $s_2 \cdot a = s_3$ . But  $s_2 \cdot b$ ,  $s_3 \cdot a$ ,  $s_3 \cdot b$ ,  $s_4 \cdot a$ , and  $s_4 \cdot b$  are not defined.

## 1.4 Augmented state space

In order to make the action  $(M(E, I), S)$  to be total, we add the state  $*$  and extend the partial maps  $(-) \cdot a : S \rightharpoonup S$  to the (total) maps  $(-) \cdot a : S \sqcup \{*\} \rightarrow S \sqcup \{*\}$  acting by  $s \cdot a = *$  if  $s \cdot a$  is not defined. Let  $S_* = S \sqcup \{*\}$  and  $* \cdot a = *$ . The pair  $(M(E, I), S_*)$  consists of a trace monoid with the total action on the set  $S_*$ . This pair is called the state space with an augmentation.

For example, the previous state space gives the augmented state space



Let  $(M(E, I), S)$  be a state space. Consider an *augmented state category*  $K_*(S)$  as follows. Its class of objects is the set  $S_* = S \sqcup \{*\}$ . Morphisms  $s \rightarrow s'$  are triples  $(s, w, s')$  of  $s \in S_*$ ,  $s' \in S_*$ ,  $w \in M(E, I)$ .

For any subset  $\Sigma \subseteq S_*$ , let  $K(\Sigma) \subseteq K_*(S)$  denotes a full subcategory with class of objects  $\Sigma$ . For  $\Sigma = S$ ,  $K(S) \subseteq K_*(S)$  will be called a *state category*.

## 1.5 Homology groups of a small category

Let  $\mathcal{C}$  be a small category and let  $F : \mathcal{C} \rightarrow \text{Ab}$  be a functor into the category of Abelian groups and homomorphisms.

**DEFINITION 1.2** Let  $\mathcal{C}$  be a small category and let  $F : \mathcal{C} \rightarrow \text{Ab}$  be a functor into the category of Abelian groups and homomorphisms. Denote by  $C_\diamond(\mathcal{C}, F)$  a chain complex of Abelian groups

$$C_n(\mathcal{C}, F) = \bigoplus_{c_0 \rightarrow \cdots \rightarrow c_n} F(c_0), \quad n \geq 0,$$

and homomorphisms  $d_n = \sum_{i=0}^n (-1)^i d_i^n : C_n(\mathcal{C}, F) \rightarrow C_{n-1}(\mathcal{C}, F)$ ,  $n > 0$ ,  
where  $d_i^n(c_0 \xrightarrow{\alpha_1} c_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} c_n, a) =$

$$\begin{cases} (c_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} c_n, F(c_0 \xrightarrow{\alpha_1} c_1)(a)) & , \quad \text{if } i = 0 \\ (c_0 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_{i-1}} c_{i-1} \xrightarrow{\alpha_{i+1}\alpha_i} c_{i+1} \xrightarrow{\alpha_{i+2}} \dots \xrightarrow{\alpha_n} c_n, a) & , \quad \text{if } 1 \leq i \leq n-1 \\ (c_0 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_{n-1}} c_{n-1}, a) & , \quad \text{if } i = n \end{cases}$$

For every integer  $n \geq 0$ , the  $n$ -th homology group  $H_n(\mathcal{C}, F)$  of  $\mathcal{C}$  with coefficients in  $F$  is the factor groups  $\text{Ker}(d_n)/\text{Im}(d_{n+1})$ .

It is well known that the functors  $H_n(C_\diamond(\mathcal{C}, -)) : \text{Ab}^\mathcal{C} \rightarrow \text{Ab}$  are isomorphic to the left derived functors  $\varinjlim_n^\mathcal{C}$  of the colimit functor  $\varinjlim^\mathcal{C} : \text{Ab}^\mathcal{C} \rightarrow \text{Ab}$ .

Hence, the Abelian groups  $H_n(\mathcal{C}, F)$  can be defined as homology groups of the complex

$$0 \leftarrow \varinjlim^\mathcal{C} P_0 \leftarrow \varinjlim^\mathcal{C} P_1 \leftarrow \varinjlim^\mathcal{C} P_2 \leftarrow \dots$$

obtained from a projective resolution

$$0 \leftarrow F \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \dots$$

of  $F \in \text{Ab}^\mathcal{C}$  by the application of the functor  $\varinjlim^\mathcal{C}$ .

## 1.6 Homology of state categories, asynchronous systems and Petri nets

For an arbitrary small category  $\mathcal{C}$ , let  $\Delta\mathbb{Z} : \mathcal{C} \rightarrow \text{Ab}$  be the functor taking constant values  $\mathbb{Z}$  at objects and  $1_\mathbb{Z} : \mathbb{Z} \rightarrow \mathbb{Z}$  at morphisms of  $\mathcal{C}$ .

By [9], an *asynchronous system* can be defined as a triple  $(S, s_0, M(E, I))$  where  $(S, M(E, I))$  is a state space and  $s_0 \in S$  is a distinguished element. Elements of  $S(s_0) = \{s \cdot \mu \mid \mu \in M(E, I)\} \subseteq S$  are *reachable states*. *Homology groups of asynchronous system with coefficients in an arbitrary functor  $F : K(S) \rightarrow \text{Ab}$*  are Abelian groups  $\varinjlim_n^{K(S(s_0))} F|_{K(S(s_0))}$ .

For a set  $B$ , denote by  $2^B$  the set of all its subsets.

A *CE net* [9] or *Petri net* [21] is a quintuple  $(B, E, pre, post, s_0)$  consisting of finite sets  $B$  and  $E$ , the maps  $pre, post : E \rightarrow 2^B$ , and a subset  $s_0 \subseteq B$ .

Let  $\mathcal{N} = (B, E, pre, post, s_0)$  be a CE net. Define an relation  $I \subseteq E \times E$  as the set of pairs  $(a, b)$  for which  $(pre(a) \cup post(a)) \cap (pre(b) \cup post(b)) = \emptyset$ .

To every element  $e \in E$  we assign a partial mapping  $(-) \cdot e : 2^B \rightharpoonup 2^B$  defined for  $s \subseteq B$  satisfying to the condition

$$(pre(e) \subseteq s) \quad \& \quad (post(e) \cap s = \emptyset).$$

In these cases, we take  $s \cdot e = (s \setminus pre(e)) \cup post(e)$  [17]. This define a partial action of  $M(E, I)$  on the set  $2^B$ . Assuming  $S = 2^B$ , we get an asynchronous system  $(S, s_0, M(E, I))$ , which corresponds to the CE net  $\mathcal{N} = (B, E, pre, post, s_0)$ . The homology groups  $H_n(\mathcal{N})$  defined as  $\varinjlim_n^{K(S(s_0))} \Delta \mathbb{Z}$  where  $S(s_0)$  is the set of reachable states.

In [9], it was built an algorithm for computing the group  $H_1(K(S), \Delta \mathbb{Z})$  and hence  $H_1(\mathcal{N})$ . It was formulated the following

**PROBLEM 1** *Construct an algorithm for computing integral homology of CE nets.*

By the definition of  $H_n(\mathcal{N})$ , this problem will be solved wenn we find an algorithm to compute the homology groups  $H_n(K(S), \Delta \mathbb{Z})$  for the state categories.

In [9], it was proved that if  $M(E, I)$  does not contain triples of pairwise independent generators, then  $H_n(K_*(S), \Delta \mathbb{Z}) = 0$  for  $n > 2$ . It was formulated

**PROBLEM 2** *Let  $n > 0$  be the maximal number of pairwise independent generators. Prove that  $H_k(K_*(S), F) = 0$  for any  $k > n$  and for any functor  $F : K_*(S) \rightarrow \text{Ab}$ .*

In the case of finite  $E$ , this conjecture proved by L. Yu. Polyakova [20]. In general solved by the author [10].

Problem 2 could not be solved for a long time. We present a way to solve this problem. Detailed proofs will be published shortly.

## 2 Semicubical sets and generalized tori

Recall the definition of semicubical set and its geometric realization. Get acquainted with generalized tori and assign to any partial trace monoid action a semicubical set.

## 2.1 Semicubical sets

Let  $\square_+$  be the category of posets  $\mathbb{I}^n$ ,  $n \in \mathbb{N}$ , where  $\mathbb{I}$  is the set  $\{0, 1\}$  ordered by  $0 < 1$ . Morphisms in  $\square_+$  are increasing maps admitting a decomposition in the composition of maps  $\delta_i^{k,\varepsilon} : \mathbb{I}^{k-1} \rightarrow \mathbb{I}^k$ ,  $1 \leq i \leq k$ ,  $\varepsilon \in \mathbb{I}$  defined as  $\delta_i^{k,\varepsilon}(x_1, \dots, x_{k-1}) = (x_1, \dots, x_{i-1}, \varepsilon, x_i, \dots, x_{k-1})$ .

A *semicubical set* is any functor  $X : \square_+^{op} \rightarrow \text{Set}$ . In [7], it is called *precubical set*. Morphisms between semicubical sets are defined as natural transformations. Any semicubical set can be given by a pair  $(X_n, \partial_i^{n,\varepsilon})$  consisting of sequence of sets  $(X_n)_{n \in \mathbb{N}}$  and a family of maps  $\partial_i^{n,\varepsilon} : X_n \rightarrow X_{n-1}$ , defined for  $1 \leq i \leq n$ ,  $\varepsilon \in \{0, 1\}$ , and satisfying to the condition

$$\partial_i^{n-1,\alpha} \circ \partial_j^{n,\beta} = \partial_{j-1}^{n-1,\beta} \circ \partial_i^{n,\alpha}, \text{ for } \alpha, \beta \in \{0, 1\}, n \geq 2 \text{ and } 1 \leq i < j \leq n.$$

These maps will be equal  $\partial_i^{k,\varepsilon} = X(\delta_i^{k,\varepsilon})$ .

Semicubical objects in an arbitrary category  $\mathcal{A}$  are defined similarly as functors  $\square_+^{op} \rightarrow \mathcal{A}$ .

## 2.2 Geometric realization

Let  $X \in \text{Set}^{\square_+^{op}}$  be a semicubical set. Its the *geometric realization* [4] is defined as the topological quotient space

$$|X|_{\square_+} = \coprod_{n \in \mathbb{N}} X_n \times [0, 1]^n / \equiv$$

with respect the smallest equivalence relation satisfying

$$(\partial_i^{n,\nu} x, t_1, \dots, t_{n-1}) \equiv (x, t_1, \dots, t_{i-1}, \nu, t_i, \dots, t_{n-1}),$$

for all  $n \geq 0$ ,  $\nu \in \{0, 1\}$ ,  $1 \leq i \leq n$ ,  $t_i \in [0, 1]$ . Geometric realization determine the functor  $|-|_{\square_+}$  assigning to every morphism of semicubical sets  $f : X \rightarrow Y$  the continuous map  $|f|_{\square_+} : |X|_{\square_+} \rightarrow |Y|_{\square_+}$  such that  $|f|_{\square_+}(x, t_1, \dots, t_n) = (f(x), t_1, \dots, t_n)$ . The functor  $|-|_{\square_+}$  can be constructed from the functor  $H : \square_+ \rightarrow \text{Top}$ ,  $H(\mathbb{I}^n) = [0, 1]^n$ , as in [5, Prop. II.1.3] by extending to the category of semicubical sets. It follows from [5, Prop. II.1.3] that  $|-|_{\square_+}$  preserves colimits.

## 2.3 Generalized tori

For a trace monoid  $M(E, I)$  with a total order relation  $<$  on  $E$ , the *generalized torus*  $T(E, I)$  is the semicubical set  $(T_n(E, I), \partial_i^{n, \varepsilon})$  such that

$$T_n(E, I) = \{(a_1, \dots, a_n) \in E^n : a_i < a_j \text{ \& } (a_i, a_j) \in I \text{ for all } 1 \leq i < j \leq n\}$$

and  $\partial_i^{n, \varepsilon}(a_1, \dots, a_n) = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$ , for all  $n \geq 0$ ,  $1 \leq i \leq n$ ,  $\varepsilon \in \{0, 1\}$ .

For example, if  $E = \{a_1, \dots, a_n\}$  ordered by  $a_1 < a_2 < \dots < a_n$  with  $I$  consisting of all pairs  $(a_i, a_j)$  for which  $i \neq j$ , then the geometric realization  $|T(E, I)|_{\square_+}$  is homeomorphic to the usual  $n$ -dimensional torus.

## 2.4 Semicubical set of a state set

Let  $(M(E, I), S)$  be a state space with a total relation  $<$  on  $E$ . Assign to it the semicubical set  $Q(E, I, S)$  with

$$Q_n(E, I, S) = \{(x, a_1, \dots, a_n) \in S_* \times T_n(E, I) \mid a_i < a_j \text{ \& } (a_i, a_j) \in I \text{ for all } 1 \leq i < j \leq n\}.$$

with the boundary maps  $\partial_i^{n, \varepsilon}(x, a_1, \dots, a_n) = (x \cdot a_i^\varepsilon, a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$  for  $1 \leq i \leq n$ ,  $n \geq 1$ ,  $\varepsilon \in \{0, 1\}$ . Here  $a^0 = 1$  and  $a^1 = a$ .

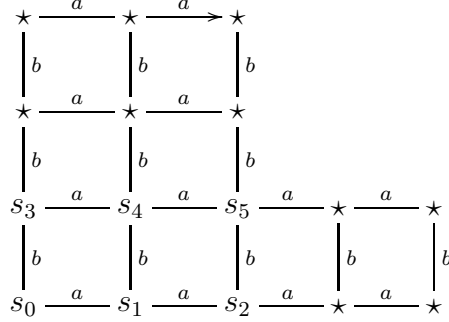
**EXAMPLE 2.1** Consider the state space consisting of  $S = \{s_0, s_1, s_2, s_3, s_4, s_5\}$ ,  $E = \{a, b\}$ ,  $I = \{(a, b), (b, a)\}$ . Elements in  $\text{Tran}$  are triples  $(s, e, s')$  corresponding to arrows  $s \xrightarrow{e} s'$  in the following diagram:

$$\begin{array}{ccccc} s_3 & \xrightarrow{a} & s_4 & \xrightarrow{a} & s_5 \\ \uparrow b & & \uparrow b & & \uparrow b \\ s_0 & \xrightarrow{a} & s_1 & \xrightarrow{a} & s_2 \end{array}$$

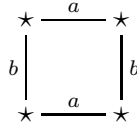
The topological space  $|Q(E, I, S)|_{\square_+}$  can be obtained from the union of



unit squares



by the identifying the vertexes  $\star$  with each other, and by identifying the segments  $\star \xrightarrow{a} \star$  with each other, and with similar identifications for the segments  $\star \xrightarrow{b} \star$  and squares



Geometric realization can be interpreted as the topological space of intermediate states of computational processes.

## 2.5 Homology groups of semicubical sets

To solve Problems 1 and 2, we need an information from the article [13].

Given a semicubical set  $X \in \text{Set}^{\square_+^{op}}$ , let  $\square_+/X$  be the category with objects  $\sigma \in \coprod_{n \in \mathbb{N}} X_n$ . Its morphisms between  $\sigma \in X_m$  and  $\tau \in X_n$  are triples  $(\alpha, \sigma, \tau)$ ,  $\alpha \in \square_+(\mathbb{I}^m, \mathbb{I}^n)$ , satisfying the relation  $X(\alpha)(\tau) = \sigma$ . *Homological system on a semicubical set*  $X$  is an arbitrary functor  $F : (\square_+/X)^{op} \rightarrow \text{Ab}$ .

Given a semicubical set  $X$  and a homological system  $F$ , consider Abelian groups  $C_n(X, F) = \bigoplus_{\sigma \in X_n} F(\sigma)$ . Let  $d_i^{n, \varepsilon} : C_n(X, F) \rightarrow C_{n-1}(X, F)$  be the homomorphisms

$$\bigoplus_{\sigma \in X_n} F(\sigma) \xrightarrow{d_i^{n, \varepsilon}} \bigoplus_{\sigma \in X_{n-1}} F(\sigma)$$

defined on the direct summands for  $1 \leq i \leq n$ ,  $\varepsilon \in \mathbb{I} = \{0, 1\}$ ,  $\sigma \in X_n$ ,  $f \in F(\sigma)$  by the equation

$$d_i^{n, \varepsilon}(\sigma, f) = (\partial_i^{n, \varepsilon}(\sigma), F(\delta_i^{n, \varepsilon}, \partial_i^{n, \varepsilon}(\sigma), \sigma)(f)).$$

For  $n \geq 0$ , the *homology groups*  $H_n(X, F)$  of *semicubical set*  $X$  with *coefficients in*  $F$  are defined as homology of the complex  $C_\diamond(X, F)$  consisting of the groups  $C_n(X, F) = \bigoplus_{\sigma \in X_n} F(\sigma)$  and differentials  $d_n = \sum_{i=1}^n (-1)^i (d_i^{n,1} - d_i^{n,0})$ . Abelian groups  $H_n(X, \Delta \mathbb{Z})$  are called to be the  $n$ th *integral homology groups*.

**Proposition 2.1** [13, Theorem 4.3] *For any semicubical set  $X$  and a homological system  $F$  on  $X$  there are isomorphisms  $\varinjlim_n^{(\square_+/X)^{op}} F \cong H_n(X, F)$ , for all  $n \geq 0$ .*

**Proposition 2.2** *For an arbitrary semicubical set  $X$  and integer  $n \geq 0$ , the group  $H_n(X, \Delta \mathbb{Z})$  is isomorphic to the  $n$ th singular homology group of the topological space  $|X|_{\square_+}$ .*

### 3 Homology of factorization category

In this section, we study and apply the Leech homology and cohomology groups of trace monoids.

#### 3.1 Factorization category

Let  $\mathcal{C}$  be a small category. Given  $\alpha \in \text{Mor } \mathcal{C}$ , denote by  $\text{cod } \alpha$  its codomain and  $\text{dom } \alpha$  the domain.

The *factorization category*  $\text{Fact}(\mathcal{C})$  has objects  $\text{Ob}(\text{Fact}(\mathcal{C})) = \text{Mor } \mathcal{C}$ , and for every  $\alpha, \beta \in \text{Mor } \mathcal{C}$  each element of  $\text{Fact}(\mathcal{C})(\alpha, \beta)$  is determined by a pair  $(f, g)$  of  $f, g \in \text{Mor } \mathcal{C}$  making commutative the diagram

$$\begin{array}{ccc} \text{cod } \alpha & \xrightarrow{g} & \text{cod } \beta \\ \alpha \uparrow & & \uparrow \beta \\ \text{dom } \alpha & \xleftarrow{f} & \text{dom } \beta \end{array}$$

For example, any monoid  $M$  considered as a small category with unique object has the factorization category  $\text{Fact}(M)$  such that  $\text{Ob}(\text{Fact}(M)) = M$ . Morphisms are given by quadruples  $\alpha \xrightarrow{(f,g)} \beta$  of  $f, \alpha, \beta, g \in M$  satisfying  $g\alpha f = \beta$ .

### 3.2 Leech homology of generalized tori

In this subsection, we present the results published in the articles [12] and [11].

*Leech homology groups of a monoid  $M$  with coefficients in a functor  $F : \text{Fact}(M)^{op} \rightarrow \text{Ab}$*  are defined as the groups  $H_n(\text{Fact}(M)^{op}, F)$ ,  $n \geq 0$ .

Given trace monoid  $M(E, I)$ , let  $\mathcal{S} : \square_+/T(E, I) \rightarrow \text{Fact}(M(E, I))$  be the functor assigning to each  $(a_1, \dots, a_n) \in \text{Ob } \square_+/T(E, I)$  the object  $a_1 \cdots a_n \in M(E, I) = \text{Ob } \text{Fact}(M(E, I))$ . Each morphism of the category  $\square_+/T(E, I)$  can be decomposed into a composition of morphisms of the form  $(\delta_i^{n, \varepsilon}), (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n), (a_1, \dots, a_n)$ . Therefore, it suffices to define  $\mathcal{S}$  on the morphisms of this kind. Let

$$\mathcal{S}(\delta_i^{n, \varepsilon}, (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n), (a_1, \dots, a_n)) = (a_1 \cdots a_{i-1} a_{i+1} \cdots a_n \xrightarrow{(a^{1-\varepsilon}, a^\varepsilon)} a_1 \cdots a_n)$$

where  $a^0 = 1$ , and  $a^1 = a$ .

**Theorem 3.1** *If  $E$  does not contain infinite subsets of pairwise independent elements, then there are natural in  $F \in \text{Ab}^{\text{Fact}(M(E, I))^{op}}$  isomorphisms*

$$H_n(\text{Fact}(M(E, I))^{op}, F) \cong H_n(T(E, I), F \circ \mathcal{S}^{op}).$$

In the case of a finite set  $E$ , this theorem allows us to construct a finite complex for computing the Leech homology groups.

### 3.3 Global dimension of a trace monoid

Cohomologies of a small categories we define by right derives of the functor  $\varprojlim_{\mathcal{C}} : \text{Ab}^{\mathcal{C}} \rightarrow \text{Ab}$ :

Let  $\mathcal{C}$  be a small category and let  $F : \mathcal{C} \rightarrow \text{Ab}$  be a functor. The category  $\text{Ab}^{\mathcal{C}}$  has enough injectives. Hence there is an injective resolution  $0 \rightarrow F \rightarrow F^0 \rightarrow F^1 \rightarrow F^2 \rightarrow \dots$ . Applying the functor  $\varprojlim_{\mathcal{C}} : \text{Ab}^{\mathcal{C}} \rightarrow \text{Ab}$  to this resolution leads to a complex

$$0 \xrightarrow{d^{-1}} \varprojlim_{\mathcal{C}} F^0 \xrightarrow{d^0} \varprojlim_{\mathcal{C}} F^1 \xrightarrow{d^1} \varprojlim_{\mathcal{C}} F^2 \rightarrow \dots$$

The  $n$ th cohomology group of  $\mathcal{C}$  with coefficients in  $F$  is defined as  $H^n(\mathcal{C}, F) = \text{Ker } d^n / \text{Im } d^{n-1}$ .

Given semicubical set  $X$  and a functor  $G : \square_+/X \rightarrow \text{Ab}$ , define *cohomology groups*  $H^n(X, G)$  of  $X$  with coefficients in  $G$  similarly to homology groups of semicubical set. Easy to see that  $H^n(X, G) \cong H^n(\square_+/X, G)$ .

The proof of [12, Theorem 2.2] contains the assertion that for each  $\alpha \in \text{Ob } \text{Fact}(M(E, I))$ ,  $H_n(\mathcal{S}/\alpha, \Delta \mathbb{Z}) = 0$  for  $n > 0$ , and  $H_0(\mathcal{S}/\alpha, \Delta \mathbb{Z}) = \mathbb{Z}$ . Hence, it follows from the Oberst Theorem [11, Prop. 1] the following assertion.

**Theorem 3.2** *For any functors  $F : \text{Fact}(M(E, I)) \rightarrow \text{Ab}$  and for all  $n \geq 0$ , there are isomorphisms  $H^n(\text{Fact}(M(E, I)), F) \cong H^n(T(E, I), F \circ \mathcal{S})$ .*

Given Abelian category  $\mathcal{A}$  its *global dimension*  $\text{gl.dim } \mathcal{A}$  is a supremum of  $n \geq 0$  for which the functors  $\text{Ext}^n(-, -)$  are not equal to 0. Theorem 3.2 leads us to the following generalization of Hilbert's Syzygy Theorem.

**Theorem 3.3** *Let  $\mathcal{A}$  be an Abelian category with coproducts and let  $M(E, I)$  be a trace monoid. If a maximal cardinality of pairwise independent elements of  $E$  equals  $n < \infty$ , then*

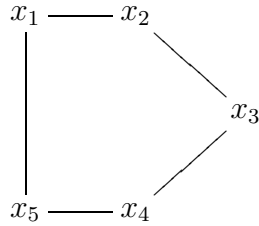
$$\text{gl.dim } \mathcal{A}^{M(E, I)} = n + \text{gl.dim } \mathcal{A}$$

*in each of the following cases:*

- (i)  $\mathcal{A}$  has exact coproducts (i.e.  $\mathcal{A}$  satisfies to the axiom AB4),
- (ii)  $\mathcal{A}$  has enough projectives.

**CONJECTURE 1** *This is true for all Abelian categories with coproducts.*

**EXAMPLE 3.1** *Let  $k$  be a field and  $E = \{x_1, x_2, x_3, x_4, x_5\}$  be the set of variables. Suppose that the independence relation  $I \subset E \times E$  is given by the following graph with vertexes  $E$  and edges  $I$ :*



Denote by  $k\langle x_1, x_2, x_3, x_4, x_5 \rangle$  the noncommutative polynomial ring in five variables. Let  $(I)$  be the ideal of  $k\langle x_1, x_2, x_3, x_4, x_5 \rangle$  generated by polynomials  $x_u x_v - x_v x_u$  for which  $(x_u, x_v) \in I$ ,  $1 \leq u, v \leq 5$ . The maximal number of independent variables equals 2. By Theorem 3.3, we have

$$\text{gl.dim } k\langle x_1, x_2, x_3, x_4, x_5 \rangle / (I) = 2.$$

### 3.4 Homology of augmented state category

Consider the functor  $\text{cod} : \text{Fact}(\mathcal{C}) \rightarrow \mathcal{C}$ ,  $\alpha \mapsto \text{cod}(\alpha)$ ,  $(\alpha \xrightarrow{(f,g)} \beta) \mapsto$ . For any  $c \in \text{Ob } \mathcal{C}$ ,  $H_n(\text{cod}/c, \Delta \mathbb{Z}) = 0$  for  $n$ .

**Proposition 3.4** *Given a small category  $\mathcal{C}$  and a functor  $F : \mathcal{C}^{op} \rightarrow \text{Ab}$ , there exists an isomorphisms  $\varinjlim_n^{\mathcal{C}^{op}} F \cong \varinjlim_n^{F_{\text{act}(\mathcal{C})^{op}}} F \circ \text{cod}^{op}$  for all  $n \geq 0$ .*

Given a state space  $(M(E, I), S_*)$  and a functor  $F : K_*(S) \rightarrow \text{Ab}$  there are isomorphisms  $H_n(K_*(S), F) \cong H_n(M(E, I)^{op}, \overline{F})$  where  $\overline{F} = \bigoplus_{x \in S_*} F(x)$

is Abelian group with the right action  $(x, f) \cdot \mu = (x\mu, F(x \xrightarrow{\mu} x\mu)(f))$ . By Proposition 3.4 and Theorem 3.1 we obtain the following complex for the computing the homology of the state space.

**Theorem 3.5** *If  $M(E, I)$  contains no infinite subsets of pairwise independent generators, then  $H_n(K_*(S), F)$  are isomorphic to  $n$ th homology groups of the complex*

$$\begin{aligned} 0 \leftarrow \bigoplus_{x \in S_*} F(x) \xleftarrow{d_1} \bigoplus_{(x, a_1) \in Q_1(E, I, S)} F(x) \xleftarrow{d_2} \bigoplus_{(x, a_1, a_2) \in Q_2(E, I, S)} F(x) \leftarrow \dots \\ \dots \leftarrow \bigoplus_{(x, a_1, \dots, a_{n-1}) \in Q_{n-1}(E, I, S)} F(x) \xleftarrow{d_n} \bigoplus_{(x, a_1, \dots, a_n) \in Q_n(E, I, S)} F(x) \leftarrow \dots, \end{aligned}$$

with differentials

$$\begin{aligned} d_n(x, a_1, \dots, a_n, f) = \\ \sum_{s=1}^n (-1)^s ((x \cdot a_s, a_1, \dots, \widehat{a_s}, \dots, a_n, F(x \xrightarrow{a_s} x \cdot a_s)(f)) \\ - (x, a_1, \dots, \widehat{a_s}, \dots, a_n, f)) \end{aligned}$$

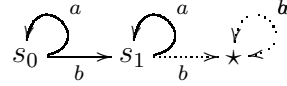
So, we have the following solution of Problem 2.

**Corollary 3.6** *If the cardinality of pairwise generators of  $M(E, I)$  not greater than  $n$ , then  $H_k(K_*(S), F) = 0$  for all  $k > n$ .*

In addition, we have a complex of finitely generated abelian groups for calculating the integral homology  $H_n(K_*(S), \Delta \mathbb{Z})$  of augmented state category.

**EXAMPLE 3.2** *Consider a state space  $\Sigma = (S, E, I, Trans)$ ,  $S = \{s_0, s_1\}$ ,  $E = \{a, b\}$ ,  $I = \{(a, b), (b, a)\}$ ,  $Tran = \{(s_0, a, s_0), (s_0, b, s_1), (s_1, a, s_1)\}$ . The set consists of two elements with the partial action of the free commutative monoid generated by  $a$  and  $b$ . Let us calculate the groups  $H_n(K_*(S), \Delta \mathbb{Z})$ .*

*We add the state  $\star$*



*and write down the matrixes of differentials. Since  $|S_*| = 3$ ,  $|Q_1(E, I, S_*)| = 6$ ,  $|Q_2(E, I, S_*)| = 3$ , the complex consists of Abelian groups*

$$0 \leftarrow \mathbb{Z}^3 \xleftarrow{d_1} \mathbb{Z}^6 \xleftarrow{d_2} \mathbb{Z}^3 \leftarrow 0$$

*The differential  $d_1(s, e) = -s \cdot e + s$  is defined by the matrix:*

$$\begin{matrix} & (s_0, a) & (s_0, b) & (s_1, a) & (s_1, b) & (*, a) & (*, b) \\ \begin{matrix} s_0 \\ s_1 \\ \star \end{matrix} & \begin{pmatrix} +1 - 1 & +1 & 0 & 0 & 0 & 0 \\ 0 & -1 & +1 - 1 & +1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 + 1 & -1 + 1 \end{pmatrix} \end{matrix}$$

*The differential  $d_2(s, e_1, e_2) = -(s * e_1, e_2) + (s, e_2) + (s * e_2, e_1) - (s, e_1)$  has the matrix:*

$$\begin{matrix} & (s_0, a, b) & (s_1, a, b) & (*, a, b) \\ \begin{matrix} (s_0, a) \\ (s_0, b) \\ (s_1, a) \\ (s_1, b) \\ (*, a) \\ (*, b) \end{matrix} & \begin{pmatrix} -1 & 0 & 0 \\ -1 + 1 & 0 & 0 \\ +1 & -1 & 0 \\ 0 & -1 + 1 & 0 \\ 0 & +1 & +1 - 1 \\ 0 & 0 & -1 + 1 \end{pmatrix} \end{matrix}$$

Using reduction of these matrices to Smith normal form, we obtain  $H_0(K_*(S), \Delta \mathbb{Z}) = \mathbb{Z}$ ,  $H_1(K_*(S), \Delta \mathbb{Z}) = \mathbb{Z}^2$ ,  $H_2(K_*(S), \Delta \mathbb{Z}) = \mathbb{Z}^1$ , and  $H_n(K_*(S), \Delta \mathbb{Z}) = 0$  for all  $n \geq 3$ .

### 3.5 Homology of Mazurkiewicz trace languages

Given  $v, w \in M(E, I)$ , we let  $v \leq w$  if there exists  $u \in M(E, I)$  such that  $vu = w$ . This relation makes  $M(E, I)$  into a partially ordered set, which we denote by  $P(E, I)$ . A *trace language* is any set of traces.

**DEFINITION 3.3** *A set  $L \subseteq M(E, I)$  is prefix closed if for all  $v \in M(E, I)$  and  $w \in L$  the relation  $v < w$  implies  $v \in L$ .*

Let  $L \subseteq M(E, I)$  be a prefix closed trace language. We have the pair  $(M(E, I), L)$  consisting of the trace monoid with the following partial action for  $v \in L$ ,  $\mu \in M(E, I)$ .

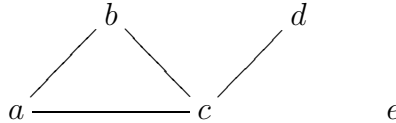
$$v \cdot \mu = \begin{cases} v\mu, & \text{if } v\mu \in L \\ \text{undefined,} & \text{otherwise.} \end{cases}$$

For any functor  $F : K_*(L) \rightarrow \text{Ab}$ , we can consider the homology groups  $H_n(K_*(L), F)$ . The groups  $H_n(K_*(L), \Delta \mathbb{Z})$  are called *integral homology groups*.

### 3.6 Homology groups of the poset of traces

Given prefix closed language  $L \subseteq M(E, I)$ , let  $\mathbb{Z}[L] : P(E, I) \rightarrow \text{Ab}$  be a functor with values  $\mathbb{Z}[L](v) = \mathbb{Z}$  for  $v \in L$  and  $\mathbb{Z}[L](v) = 0$ , otherwise. For  $u \leq v \in L$ , we will define  $\mathbb{Z}[L](u \leq v) = 1_{\mathbb{Z}}$ . We study the homology groups  $H_n(P(E, I), \mathbb{Z}[L])$  of the poset  $P(E, I)$  and their relationship with  $H_n(K_*(L), \Delta \mathbb{Z})$ .

Denote by  $p_n$  the cardinality of the set of  $n$ -cliques in the graph  $(E, I)$ . In particular,  $p_0 = 1$  as the number of empty subsets in  $E$ ,  $p_1 = |E|$ . For example, if  $(E, I)$  is the graph



then  $p_0 = 1$ ,  $p_1 = 5$ ,  $p_2 = 4$ ,  $p_3 = 1$ .

**Theorem 3.7**  $H_n(K_*(L), \Delta \mathbb{Z}) \cong H_n(P(E, I), \mathbb{Z}[L]) \oplus \mathbb{Z}^{(p_n)}.$

Given a partially ordered set  $P$ , let  $\widetilde{H}_n(P)$  be the reduced singular homology of the classifying space  $B(P)$ . It is not hard to see that  $H_n(P(E, I), \mathbb{Z}[L]) \cong \widetilde{H}_{n-1}(P(E, I) \setminus L)$  for  $n \geq 1$ .

**Corollary 3.8**  $H_n(K_*(L), \Delta \mathbb{Z}) \cong \widetilde{H}_{n-1}(P(E, I) \setminus L) \oplus \mathbb{Z}^{(p_n)}$  for all  $n \geq 1$ .

We see that  $H_1(K_*(L), \Delta \mathbb{Z})$  is a free Abelian group.

**CONJECTURE 2** For any trace monoid  $M(E, I)$  with partial action on a set  $S$ , the Abelian group  $H_1(K_*(S), \Delta \mathbb{Z})$  is free.

Note the following homological properties of partially ordered set of traces. We assume that the language of traces  $L$  is prefix closed.

- If  $I = \{(a, b) \in E \times E \mid a \neq b\}$  and hence  $M(E, I)$  is commutative, then  $H_n(P(E, I), \mathbb{Z}[L]) = 0$  for all  $n \geq 1$ .
- If  $I = \emptyset$  and hence  $M(E, I)$  is free, then  $H_n(P(E, I), \mathbb{Z}[L]) = 0$  for all  $n \geq 2$ .
- For arbitrary finitely generated Abelian groups  $A_1, A_2, \dots, A_n$  with free  $A_1$ , there exists a trace monoid  $M(E, I)$  such that  $H_n(P(E, I), \mathbb{Z}[\{1\}]) \cong A_k$  for all  $1 \leq k \leq n$ .

### 3.7 Baues-Wirsching homology of the state category

Let  $M(E, I)$  be an arbitrary trace monoid and let  $X$  be a right  $M(E, I)$ -set. Recall that  $K(X)$  denotes the category of states with objects  $x \in X$  and morphisms  $x \xrightarrow{\mu} x\mu$  for  $x \in X$  and  $\mu \in M(E, I)$ . Considering  $M(E, I)$  as a category with an unique object we can define a functor  $U : K(X) \rightarrow M(E, I)$  assigning to each morphism  $x \xrightarrow{\mu} x\mu$  the morphism  $\mu \in M(E, I)$ . Applying the functor  $Fact$  to  $U$ , we can consider a functor  $Fact(U) : Fact(K(X)) \rightarrow Fact(M(E, I))$ . For any functor  $F : Fact(K(X))^{op} \rightarrow \text{Ab}$ , there exists its Kan extension  $Lan^{Fact(U)^{op}} : Fact(K(M(E, I)))^{op} \rightarrow \text{Ab}$  [16].

**Theorem 3.9** Given functor  $F : Fact(K(X))^{op} \rightarrow \text{Ab}$ , there exist isomorphisms

$$H_n(Fact(K(X))^{op}, F) \cong H_n(Fact(M(E, I))^{op}, Lan^{Fact(U)^{op}} F)$$

for all  $n \geq 0$ .



### 3.8 The solution of Problem 1

Recall that a state space  $(M(E, I), S)$  is a trace monoid with a partial action on  $S$ . The category of states  $K(S) \subset K_*(S)$  is the full subcategory with objects  $s \in S$ . Denote by  $\mathbb{Z}S$  the free Abelian group generated by  $s \in S$ . Let  $\overline{Q}_n(E, I, S) = \{(s, a_1, \dots, a_n) \in S \times T_n(E, I) | sa_1 \cdots a_n \neq \star\}$ .

**Theorem 3.10** *Given a state space  $(M(E, I), S)$ , the groups  $H_n(K(S), \Delta \mathbb{Z})$  are isomorphic to the homology groups of the complex*

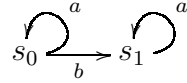
$$\begin{aligned} 0 \leftarrow \mathbb{Z}(S) \xleftarrow{d_1} \mathbb{Z}\overline{Q}_1(S, E, I) \xleftarrow{d_2} \mathbb{Z}\overline{Q}_2(S, E, I) \leftarrow \cdots \\ \cdots \leftarrow \mathbb{Z}\overline{Q}_{n-1}(S, E, I) \xleftarrow{d_n} \mathbb{Z}\overline{Q}_n(S, E, I) \leftarrow \cdots \end{aligned}$$

with differentials

$$\begin{aligned} d_n(s, a_1, \dots, a_n) = \sum_{i=1}^n (-1)^i (sa_i, a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n) \\ - \sum_{i=1}^n (-1)^i (s, a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n) \end{aligned}$$

Consider an example of computing the homomology groups of a state category.

**EXAMPLE 3.4** *Let  $M(E, I)$  be a commutative trace monoid generated by two elements and let  $S$  consists of two elements. That is  $E = \{a, b\}$ ,  $I = \{(a, b), (b, a)\}$ ,  $S = \{s_0, s_1\}$ . The generators act by  $s_0a = s_0$ ,  $s_0b = s_1$ ,  $s_1a = s_1$  as it is shown in the following picture.*



The complex consists of abelian groups

$$C_0 = \mathbb{Z}\{s_0, s_1\}, \quad C_1 = \mathbb{Z}\{(s_0, a), (s_0, b), (s_1, a)\}, \quad C_2 = \mathbb{Z}\{(s_0, a, b)\}.$$

We have a complex  $0 \leftarrow \mathbb{Z}^2 \xleftarrow{d_1} \mathbb{Z}^3 \xleftarrow{d_2} \mathbb{Z} \leftarrow 0 \leftarrow 0 \leftarrow \cdots$ . The differential  $d_1$  is described by the following matrix.

$$\begin{array}{c} (s_0, a) \quad (s_0, b) \quad (s_1, a) \\ s_0 \left( \begin{array}{ccc} 1 & -1 & 1 \\ 0 & -1 & 1 \end{array} \right) \\ s_1 \end{array}$$

The differential  $d_2$  has the following matrix.

$$\begin{array}{c} (s_0, a, b) \\ (s_0, a) \left( \begin{array}{c} -1 \\ -1 + 1 \\ +1 \end{array} \right) \\ (s_0, b) \\ (s_1, a) \end{array}$$

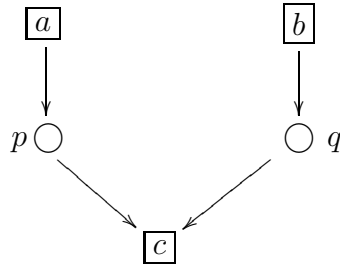
Using the reduction to the Smith normal forms, we get

$$H_0(K(S), \Delta \mathbb{Z}) = \mathbb{Z}, H_1(K(S), \Delta \mathbb{Z}) = \mathbb{Z}, H_n(K(S), \Delta \mathbb{Z}) = 0 \text{ for all } n \geq 2.$$

### 3.9 Homology groups of CE nets

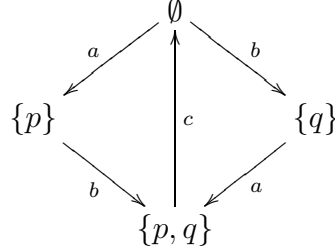
For the computing the homology groups of a finite CE net, we first construct the state space  $(M(E, I), S(s_0))$ . Then we can compute  $H_n(K(S(s_0)), \Delta \mathbb{Z})$  by the method described above.

Let, for example,  $\mathcal{N}$  be the following CE net.



The corresponding trace monoid  $M(E, I)$  is defined by  $E = \{a, b, c\}$  and  $I = \{(a, b), (b, a)\}$ . The set of states  $S$  consists of all subsets  $s \subseteq \{p, q\}$ . The corresponding asynchronous system  $(M(E, I), S, s_0)$  is defined by  $s_0 = \emptyset$  and

a partial action of  $M(E, I)$  shown in the following figure.



That is  $\emptyset \cdot a = \{p\}$ ,  $\emptyset \cdot b = \{q\}$ ,  $\{p\} \cdot b = \{p, q\}$ ,  $\{q\} \cdot a = \{p, q\}$ , and  $\{p, q\} \cdot c = \emptyset$ . All states are admissible. Hence  $S(s_0) = S$ . The complex consists of the Abelian groups

$$\begin{aligned}
C_0 &= \mathbb{Z}\{\emptyset, \{p\}, \{q\}, \{p, q\}\} \cong \mathbb{Z}^4, \\
C_1 &= \mathbb{Z}\{(\emptyset, a), (\emptyset, b), (\{p\}, b), (\{q\}, a), (\{p, q\}, c)\} \cong \mathbb{Z}^5, \\
C_2 &= \mathbb{Z}\{(\emptyset, a, b)\} \cong \mathbb{Z}.
\end{aligned}$$

The differential  $d_1(s, e) = -s \cdot e + s$  has the following matrix.

$$\begin{array}{c}
\emptyset \\
\{p\} \\
\{q\} \\
\{p, q\}
\end{array}
\begin{pmatrix}
& (\emptyset, a) & (\emptyset, b) & (\{p\}, b) & (\{q\}, a) & (\{p, q\}, c) \\
1 & 1 & 0 & 0 & 0 \\
-1 & 0 & 1 & 1 & -1 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 1
\end{pmatrix}$$

We have  $d_2(\emptyset, a, b) = -(\emptyset \cdot a, b) + (\emptyset, b) + (\emptyset \cdot b, a) - (\emptyset, a)$ . Hence, the matrix of  $d_2$  is described by the matrix

$$\begin{array}{c}
(\emptyset, a, b) \\
(\emptyset, a) \\
(\emptyset, b) \\
(\{p\}, b) \\
(\{q\}, a) \\
(\{p, q\}, c)
\end{array}
\begin{pmatrix}
-1 \\
1 \\
-1 \\
1 \\
0
\end{pmatrix}$$

We have the following complex for the computing  $H_n(\mathcal{N})$  for all  $n \geq 0$ .

$$0 \leftarrow \mathbb{Z}^4 \xleftarrow{d_1} \mathbb{Z}^5 \xleftarrow{d_2} \mathbb{Z} \leftarrow 0 \leftarrow 0 \leftarrow \dots$$

Using the Smith normal forms, we get  $H_0(\mathcal{N}) = \mathbb{Z}$ ,  $H_1(\mathcal{N}) = \mathbb{Z}$ , and  $H_n(\mathcal{N}) = 0$ , for all  $n \geq 2$ .

## 4 Conclusion

The author believes that the results will help in investigation the Goubault homology of asynchronous systems as the homology groups  $H_n(K(S), \mathbb{Z}^\varepsilon)$ ,  $\varepsilon \in \{0, 1\}$ , with coefficients in some suitable systems of Abelian groups. You can explore the  $n$ -deadlocks for asynchronous systems. It is possible to find homological signs for the existence of bisimilar equivalence between asynchronous systems, Petri nets, and trace languages.

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